

Quasi-exact minus-quartic oscillators in strong-core regime

Miloslav Znojil

OTF, Ústav jaderné fyziky AV ČR,
250 68 Řež, Czech Republic
email: znojil@ujf.cas.cz

Abstract

\mathcal{PT} -symmetric potentials $V(x) = -x^4 + iBx^3 + Cx^2 + iDx + iF/x + G/x^2$ are quasi-exactly solvable, i.e., a specific choice of a small $G = G^{(QES)} = \text{integer}/4$ is known to lead to wave functions $\psi^{(QES)}(x)$ in closed form at certain charges $F = F^{(QES)}$ and energies $E = E^{(QES)}$. The existence of an alternative, simpler and non-numerical version of such a construction is announced here in the new dynamical regime of very large $G^{(QES)} \rightarrow \infty$.

PACS

03.65.Ca; 03.65.Ge;

1 Introduction

Eight years ago Bender and Boettcher [1] considered a specific \mathcal{PT} -symmetric “asymptotically repulsive” oscillator

$$H = p^2 + V^{(BB)}(x), \quad V^{(BB)}(x) = -x^4 + 2iax^3 + (a^2 - 2b)x^2 + 2i(ab - N)x \quad (1)$$

and conjectured and numerically verified that it possesses the real and discrete spectrum in certain intervals of couplings a and b (note that while \mathcal{P} denotes the operator of parity, the complex conjugation \mathcal{T} mimics time reversal so that a and b must be chosen real). These authors emphasized that their non-Hermitian model may be considered, in a way, a “nearest neighbor” of the harmonic oscillator as it exhibits, in a sharp contrast to its undeservedly more popular Hermitian and asymptotically growing $+x^4$ alternative [2], the exceptional quasi-exact solvability (QES, [3]).

By definition the latter feature means that in a way which parallels harmonic-oscillator wave functions $\psi(\vec{x}) \sim \exp(-\vec{x}^2/2) \times a \text{ polynomial}$, a part of the set of the bound states generated by the Hamiltonian (1) remains elementary,

$$\psi^{(BB)}(x) = e^{-ix^3/3 - ax^2/2 - ibx} \sum_{k=0}^N c_k x^k. \quad (2)$$

This observation acquires a particular appeal in the light of the recent increase of interest in the possible applications of non-Hermitian models in quantum optics [4] and in the analyses of quantum chaos [5] as well as in various innovations of supersymmetric [6], magnetohydrodynamical [7] or particle-physics [8] models. During the recent quick development of the related theory of \mathcal{PT} -symmetric models [9] - [11] it has been, moreover, revealed that their quantum bound states may be assigned the standard probabilistic interpretation, provided only that one re-defines the scalar product in Hilbert space, $\langle \cdot | \cdot \rangle_{Dirac} \rightarrow \langle \cdot | \cdot \rangle_{adapted}$. For this purpose one only has to introduce an unusual, Hamiltonian-dependent metric operator $\Theta \neq I$ in a way which proved productive in nuclear physics [12],

$$\langle \psi_1 | \psi_2 \rangle_{Dirac} \longrightarrow \langle \psi_1 | \psi_2 \rangle_{adapted} \equiv \langle \psi_1 | \Theta | \psi_2 \rangle_{Dirac}. \quad (3)$$

It is now agreed [13] that the \mathcal{PT} -symmetric Hamiltonians $H^{(\mathcal{PT}-symmetric)}$ may be used as phenomenological models whenever we succeed in an explicit construction of the metric operator $\Theta = \Theta(H)^1$.

The latter observations enhance the importance of the Bender’s and Boettcher’s partially solvable two-parametric model (1) as well as of its straightforward three-parametric “charged” and “spiked” generalization, with a Coulomb and centrifugal force added in ref. [14],

$$V(x) = -x^4 + iBx^3 + Cx^2 + iDx + iF/x + G/x^2. \quad (4)$$

Unfortunately, the phenomenological applicability of both these models proved unexpectedly hindered by the computational difficulties arising during the explicit construction of their exact bound states (cf. section 2 for a brief review). In a reaction to such a contradictory situation we returned

¹See, e.g., the September special issue of Czech. J. Phys. 55 (2005), pp. 1045 - 1192 for more details.

to this class of models once more. We revealed and report here a significant simplification of the QES construction which emerges in a strongly spiked limit, i.e., for very large couplings $G \rightarrow \infty$.

In full detail our observations will be described in section 3 and summarized in section 4 emphasizing that the new dynamical regime is complementary to the two-parametric option (1) of ref. [1] with vanishing G . Our new construction may even be considered simpler since it leads to the determination of the negative-quartic \mathcal{PT} -symmetric QES bound states in terms of closed and compact formulae.

2 Quartic models and their quasi-exact solvability

2.1 A broader family of the next-to-harmonic models?

Among all the available exactly solvable versions of Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ in Quantum Mechanics, an undoubtedly exceptional position belongs to the harmonic oscillator, the Hamiltonian of which preserves *the same* differential-operator form in *both* the \vec{x} - and \vec{p} -representations [15]. Although such a curious “Fourier-transformation-symmetry” property of $H^{(HO)} = \vec{p}^2 + \vec{x}^2$ does not survive the transition to the “next”, quartic anharmonic oscillators, it still may play a role in their perturbative [16] or continued-fraction [17] description. Moreover, an unexpected role of the Fourier-transformation partnership between two *different* quartic oscillators has been revealed by Buslaev and Grecchi who succeeded in proving a strict isospectrality between certain two “next-to-harmonic” quartic-oscillator models $H^{(Hermitian)}$ and $H^{(\mathcal{PT}-symmetric)}$ (cf. [18]).

The subsequent increase of interest in \mathcal{PT} -symmetry in Quantum Mechanics [9] climaxed recently, in the specific quartic-oscillator context, with the paper [19] where, for a sample choice of the negative-quartic $H^{(\mathcal{PT}-symmetric)} \sim -x^4$, an explicit construction of the metric Θ has been presented as performed without *ad hoc* tricks and starting simply from the first principles. The related Buslaev’s and Grecchi’s results are recollected there so that, in some sense, the circle is closed and the picture seems completed. Yet, the description of another, viz., QES harmonic-oscillator-like property of models $H^{(\mathcal{PT}-symmetric)}$ deserves an independent completion.

In a way indicated by Buslaev and Grecchi ([18], cf. also [20]) and re-emphasized, e.g., by Dorey et al [21], our understanding of the various aspects of \mathcal{PT} -symmetry may be made simpler rather than more complicated by an introduction of the angular momentum L in our ordinary Schrödinger equation,

$$\left[-\frac{d^2}{dx^2} + \frac{L(L+1)}{x^2} + V(x) \right] \psi(x) = E\psi(x). \quad (5)$$

Traditionally one abbreviates

$$L = \frac{1}{2}(d-3), \ 1 + \frac{1}{2}(d-3), \ 2 + \frac{1}{2}(d-3), \ \dots \quad (6)$$

in $d \geq 3$ dimensions [18] but one may also take into consideration the centrifugal-like spike in the potential (4). Thus, a generalization (1) \rightarrow (4) is to be understood as a transition to the singular models with $F \neq 0$ and/or with

$$L(L+1) + G = \ell(\ell+1) \neq 0, \quad \ell = \sqrt{G + \left(L + \frac{1}{2}\right)^2} - \frac{1}{2}. \quad (7)$$

In ref. [14], a theoretical merit of such a step has been seen in the identification of the older regular model (1) with the mere *special case* of eq. (4). Indeed, the vanishing of the charge $F = F^{(QES)}$ as *postulated* in ref. [1] *results*, in fact, directly from the QES conditions at $\ell(\ell + 1) = 0$.

Another consequence of the formal presence of the centrifugal term in eq. (5) lies in the related possibility of a modification of the potential (i.e., of the dynamics) by a mere formal change of the variables in eq. (5) [20]. This idea will not be discussed here in any detail but the interested reader may consult ref. [22] for an illustration.

2.2 $|x| \gg 1$ asymptotics for the decreasing quartic potentials

The general QES recipe starting from a polynomial potential [say, (1) or (4)] constructs its QES bound states [exemplified here by eq. (2)] in a way described by Magyari [23]. Basically, the construction parallels the harmonic-oscillator factorization $\psi(\vec{x}) \sim \exp(-\vec{x}^2/2) \times$ a *polynomial* where, for the \mathcal{PT} -symmetric quartic model (4) with five real couplings, one extracts and separates the $|x| \gg 1$ asymptotically dominant part of the normalizable (i.e., bound-state) wave function into its exponential factor,

$$\psi(x) = \exp\left(-\frac{1}{3}ix^3 - \frac{1}{2}\beta x^2 - i\gamma x\right) \sum_{n=0}^{\infty} \omega_n (ix)^{n+p} \quad \beta = B/2, \quad \gamma = (\beta^2 - C)/2. \quad (8)$$

In such a scenario and in a way extending eq. (2) to $\ell \neq 0$, all the QES states will be characterized by the *exact* reduction of the infinite series to a polynomial,

$$\omega_{N+1} = \omega_{N+2} = \dots = 0. \quad (9)$$

This means that our Schrödinger eq. (5) must be integrated over a complex contour of coordinates $x \in \mathcal{C}$ which is bent downwards, say, towards its asymptotes

$$\mathcal{C}_{left} \sim -\varrho e^{+i\varphi_{left}}, \quad \mathcal{C}_{right} \sim +\varrho e^{-i\varphi_{right}}, \quad 0 < \varphi_{left, right} < \frac{\pi}{3} \quad (10)$$

with the large and positive real parameter $\varrho \rightarrow \infty$. Indeed, it is easy to verify that the exponent in eq. (8) decreases along both these half-lines,

$$e^{-\frac{1}{3}ix_{left}^3} = e^{+\frac{1}{3}i\varrho^3(\cos 3\varphi_{left} + i \sin 3\varphi_{left})} \approx e^{-\frac{1}{3}\varrho^3 \sin \varphi_{left}} \times \text{a bounded oscillatory factor}, \quad (11)$$

$$e^{-\frac{1}{3}ix_{right}^3} = e^{-\frac{1}{3}i\varrho^3(\cos 3\varphi_{right} - i \sin 3\varphi_{right})} \approx e^{-\frac{1}{3}\varrho^3 \sin \varphi_{right}} \times \text{a bounded oscillatory factor}. \quad (12)$$

As long as our problem is analytic in the whole cut complex plane (with the cut starting in the origin and oriented upwards), the contour \mathcal{C} may be chosen as safely avoiding the singularity in the origin. Thus, in terms of the effective angular momentum ℓ we have to fix the sub-exponential exponent $p \equiv -\ell$ in eq. (8) in a way compatible with both refs. [1] and [14].

2.3 Magyari's QES conditions

The insertion of the ansatz (8) + (9) in Schrödinger equation (5) fixes the QES-compatible value of the coupling at the linear potential term,

$$D = D(N) = 2(\ell + \beta\gamma - N - 1) \quad (13)$$

and imposes, furthermore, the following overcomplete linear algebraic set of $N + 2$ constraints upon the $N + 1$ (arbitrarily normalized) wave function coefficients ω_n ,

$$(2\ell - n)(n + 1)\omega_{n+1} + [F - 2\gamma(\ell - n)]\omega_n + [E - \gamma^2 + \beta(2\ell - 2n + 1)]\omega_{n-1} + 2(N + 2 - n)\omega_{n-2} = 0 \quad (14)$$

where n runs from 0 till $N - 1$. With obvious abbreviations, these equations may be re-written as a non-square matrix problem

$$\begin{pmatrix} S_0(F) & U_0 & & & & & \\ T_1(E) & S_1(F) & U_1 & & & & \\ W_2 & T_2(E) & S_2(F) & U_2 & & & \\ & W_3 & T_3(E) & S_3(F) & U_3 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & W_{N-1} & T_{N-1}(E) & S_{N-1}(F) & U_{N-1} \\ & & & & W_N & T_N(E) & S_N(F) \\ & & & & & W_{N+1} & T_{N+1}(E) \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_N \end{pmatrix} = 0. \quad (15)$$

It must be solved numerically in general [24].

3 Two feasible versions of the QES construction

3.1 The domain of the small G , L and ℓ

In the practical computations one may treat eq. (15) as the two linear square-matrix eigenvalue problems

$$\begin{pmatrix} S_0(0) & U_0 & & & & & \\ T_1(E) & S_1(0) & U_1 & & & & \\ W_2 & T_2(E) & S_2(0) & U_2 & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & W_{N-1} & T_{N-1}(E) & S_{N-1}(0) & U_{N-1} & \\ & & & W_N & T_N(E) & S_N(0) \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_N \end{pmatrix} = -F_e(E) \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_N \end{pmatrix} \quad (16)$$

$$\begin{pmatrix} T_1(0) & S_1(F) & U_1 & & & & \\ W_2 & T_2(0) & S_2(F) & U_2 & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & W_{N-1} & T_{N-1}(0) & S_{N-1}(F) & U_{N-1} & \\ & & & W_N & T_N(0) & S_N(F) & \\ & & & & W_{N+1} & T_{N+1}(0) \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_N \end{pmatrix} = -E_e(F) \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_N \end{pmatrix} \quad (17)$$

which are complemented by the two mutual-coupling conditions

$$E = E_e(F), \quad F = F_e(E). \quad (18)$$

An important note is to be added, based on the inspection of eq. (14). It reveals that in the Magyari's non-square matrix constraint (15), the upper-diagonal coefficient $U_n = (2\ell - n)(n + 1)$

can in fact vanish at $n = n(\ell) = 2\ell$, i.e., at all the half-integer effective angular momenta ℓ . In eq. (5) this represents a constraint upon the freedom in the choice of the spike strength G . Thus, whenever the integer 2ℓ does not exceed the dimension N , at least one of the coupled secular determinants factorizes [14] and a significant reduction of the complexity of the algebraic QES conditions is achieved. Still, the difficulties with the construction grow fairly quickly with the growth of 2ℓ even at the integer values of this parameter.

In the light of the latter comment one may be quite surprised by our forthcoming present main result saying that a *dramatic and drastic* simplification of the recipe recurs in the asymptotic domain where $\ell \rightarrow \infty$.

3.2 Quartic QES models and their unexpected duality at the large ℓ

A few *non-numerical* samples of the solution of eqs. (15) may be found in refs. [1] (using $\ell = F = G = 0$ and several small N) and [14] (using $\ell = 1/2$ and N up to 4, or $\ell = 1$ and N up to 3). Also the results of these studies confirm that serious computational difficulties arise and grow very quickly whenever ℓ grows beyond one. In parallel [24], the form of eq. (15) appears to be perceivably simpler whenever *all* the values of the *other* parameters N , β and γ become negligible in comparison with the partial-wave index L and/or with the strength of the core G . In the latter dynamical regime where $\ell \gg \max(N, |\beta|, |\gamma|)$ we may omit the negligible terms from our eq. (15) and get the leading-order version of the QES requirement,

$$\begin{pmatrix} F - 2\gamma\ell & 2\ell & & & & \\ E + 2\beta\ell & F - 2\gamma\ell & 4\ell & & & \\ 2N & E + 2\beta\ell & F - 2\gamma\ell & 6\ell & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 6 & E + 2\beta\ell & F - 2\gamma\ell & 2N\ell \\ & & & 4 & E + 2\beta\ell & F - 2\gamma\ell \\ & & & & 2 & E + 2\beta\ell \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_N \end{pmatrix} = 0. \quad (19)$$

In this equation we may re-scale the coefficients $\omega_n = h_n \ell^{-n/3}$ and subtract the leading-order asymptotic approximants,

$$F = 2\gamma\ell + 2s\ell^{2/3}, \quad E = -2\beta\ell + 2t\ell^{1/3}. \quad (20)$$

This replaces eq. (19) by its strictly equivalent but strikingly simpler form

$$\begin{pmatrix} s & 1 & & & & \\ t & s & 2 & & & \\ N & t & s & 3 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 3 & t & s & N \\ & & & 2 & t & s \\ & & & & 1 & t \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_N \end{pmatrix} = 0. \quad (21)$$

The phenomenologically most important and formally most remarkable consequence of this result is that it represents another manifestation of the Buslaev's and Grecchi's [18] duality between

Hermitian and non-Hermitian quartic oscillators, this time on the level of their respective QES subsets. Indeed, in the light of ref. [25], *the same equation played the same role* for the *Hermitian* asymptotically growing potentials

$$V^{(Hermitian)}(r) = +r^4 + B r^3 + C r^2 + D r + F/r + G/r^2, \quad r \in (0, \infty). \quad (22)$$

Amazingly enough, all the numerous differences between the potentials (4) and (22) (the latter being defined on the half-axis of course) disappear on the level of constraint (21). This enables to make the rest of our present text short. We may just cite the final (though, by the way, not so easily derived!) results of the extensive computations as performed in ref. [25]. In particular, this enables us to summarize that the real roots $s = t = t(N)$ of eq. (21) form the N -dependent and equidistant multiplets of integers,

$$t(N) = t_k(N) = N - 3k, \quad k = 0, 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor. \quad (23)$$

This means that the physically acceptable solutions of our present \mathcal{PT} -symmetric $\ell \gg 1$ QES problem exist and occur in the multiplets with the following asymptotic energies and charges,

$$E = -2\beta\ell + 2(N - 3k)\ell^{1/3} + \dots, \quad F = 2\gamma\ell + 2(N - 3k)\ell^{2/3} + \dots, \quad k = 0, 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor. \quad (24)$$

One may now return to the elementary recurrences (21) and evaluate, very easily, the coefficients ω_n of the wave functions in the same next-to-leading order approximation. In the light of the existing thorough analysis of this problem in the dual Hermitian context [25], this task may already be left to the readers as an exercise.

4 Summary

Could we view the quasi-exactly solvable \mathcal{PT} -symmetric quartic potentials as a choice, in some sense, “next” to the popular harmonic oscillator? In our paper we tried to support an affirmative answer.

During our study we felt particularly motivated by the technical difficulties arising in connection with the explicit construction of the quartic QES charges. Although, implicitly, they are defined by the coupled pair of the Magyari’s polynomial algebraic equations for two unknowns, their practical determination must usually rely upon the computerized, Gröbner-basis-based algebraic manipulation techniques and numerical root-searching [26]. In addition, it is quite unpleasant that the complexity of the latter algorithm grows fairly quickly with the growth of the degree N of the polynomial wave functions as well as with the growth of the angular momentum ℓ . Finally, the situation significantly worsens whenever ℓ ceases to be a half-integer [14].

We were encouraged by the well known fact that, quite often, the dependence of bound state on the angular momentum may get simplified in an asymptotic regime [27]. In the latter direction, our attempt proved successful. We found that several large- ℓ properties of our non-Hermitian model (like, e.g., the subtle QES-related cancellations of the separate elements in the infinite power series in x) find in fact quite close parallels in its self-adjoint predecessors. Many differences (e.g., the occurrence of complex couplings or the deformability of the integration contours in \mathcal{PT} -symmetric

case) proved inessential. We arrived at the final version (21) of the Magyari's equation which is, from the formal point of view, identical with the equations encountered in Hermitian cases (so, we could also employ their well known solutions in our construction).

In conclusion we may add that whenever necessary, one may leave the asymptotic domain and switch attention to the finite effective angular momenta $\ell \ll \infty$. The necessary mathematics may be found in the modified Rayleigh-Schrödinger perturbation recipe as adapted to non-square matrices in ref. [28]. It is worth emphasizing that it makes the full use of the finite-dimensional character of the Magyari's re-formulation of our present negative-quartic QES Schrödinger equation. For this reason it may be recommended as an efficient and systematic source of higher-order corrections.

Acknowledgment

The research was partially supported by the projects LC06002 and IRP AV0Z10480505.

References

- [1] C. M. Bender and S. Boettcher, J. Phys. A: Math. Gen. 31 (1998) L273.
- [2] A. G. Ushveridze, Quasi-exactly solvable models in quantum mechanics. IOPP, Bristol, 1994.
- [3] A. Hautot, Phys. Lett. A 38 (1972) 305;
V. Singh, V., S. N. Biswas and K. Datta, Phys. Rev. D 18 (1978) 1901;
G. P. Flessas and A. Watt, J. Phys. A: Math. Gen. 14 (1981) L315;
M. Znojil, J. Phys. A: Math. Gen. 15 (1982) 2111;
A. V. Turbiner, Comm. Math. Phys. 118 (1988) 467;
M. Znojil and P. G. L. Leach, J. Math. Phys. 33 (1992) 2785;
D. Gómez-Ullate, A. González-López and M. A. Rodríguez, Res. Expo. Math. 25 (2002) 211.
- [4] M. V. Berry, Czech. J. Phys. 54 (2004) 1039.
- [5] A. Nanayakkara, Phys. Lett. A 334 (2005) 144;
H. Břila, M. Tater and M. Znojil, Phys. Lett. A 351 (2006) 452.
- [6] G. Lévai, Czech. J. Phys. 54 (2004) 1121
- [7] U. Günther and F. Stefani, Czech. J. Phys. 55 (2005) 1099.
- [8] F. Kleefeld, Czech. J. Phys. 55 (2005) 1123.
- [9] C. M. Bender and K. A. Milton, Phys. Rev. D 55 (1997) R3255;
C. M. Bender, S. Boettcher and P. N. Meissinger, J. Math. Phys. 40 (1999) 2201;
A. Mostafazadeh, J. Math. Phys. 43 (2002) 205.
- [10] F. M. Fernández, R. Guardiola, J. Ros and M. Znojil, J. Phys. A: Math. Gen. 31 (1998) 10105;
B. Bagchi, S. Mallik and C. Quesne, Mod. Phys. Lett. A17 (2002) 1651;
R. Kretschmer and L. Szymanowski, Czech. J. Phys. 54 (2004) 71;
H. Langer and Ch. Tretter, Czech. J. Phys. 54 (2004) 1113.
- [11] A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. 37 (2004) 11645;
A. Mostafazadeh, Czech. J. Phys. 54 (2004) 1125;
E. Caliceti, S. Graffi and J. Sjöstrand, J. Phys. A: Math. Gen. 38 (2005) 185.
- [12] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.
- [13] C. M. Bender, Czech. J. Phys. 54 (2004) 13 and 1027;
A. Mostafazadeh, Czech. J. Phys. 55 (2005) 1157.
- [14] M. Znojil, J. Phys. A: Math. Gen. 33 (2000) 4203.

- [15] S. Flügge, Practical Quantum Mechanics I. Springer, New York, 1971, p. 178.
- [16] I. G. Halliday and P. Suranyi, Phys. Rev. D 21 (1980) 1529.
- [17] M. Znojil, Phys. Rev. D 24 (1981) 903.
- [18] V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. 26 (1993) 5541.
- [19] H. F. Jones and J. Mateo, An Equivalent Hermitian Hamiltonian for the non-Hermitian $-x^4$ Potential. LANL arXiv quant-ph/0601188.
- [20] M. Znojil, J. Phys. A: Math. Gen. 27 (1994) 4945.
- [21] P. Dorey, C. Dunning and R. Tateo, J. Phys. A: Math. Gen. 34 (2001) 5679.
- [22] M. Znojil, J. Phys. A: Math. Gen. 33 (2000) 6825.
- [23] E. Magyari, Phys. Lett. A 81 (1981) 116.
- [24] M. Znojil, Anharmonic oscillator in the new perturbative picture (JINR Dubna, Russia, 1989), Communication Nr. E5 - 89 - 726.
- [25] M. Znojil, J. Math. Chem. 26 (1999) 157.
- [26] M. Znojil, D. Yanovich and V. P. Gerdt, J. Phys. A: Math. Gen. 36 (2003) 6531.
- [27] Y. P. Varshni, Phys. Rev. A 40 (1989) 2180;
M. Znojil, F. Gemperle and O. Mustafa, J. Phys. A: Math. Gen. 35 (2002) 5781;
N. Saad, R. L. Hall and A. B. von Keviczky, J. Phys. A: Math. Gen. 36 (2003) 487;
H. Bíla, Czech. J. Phys. 54 (2004) 1049.
- [28] M. Znojil, Phys. Lett. A 341 (2005) 67.